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Conifold holography

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Abstract: We examine the extension of the Klebanov-Witten gauge/gravity correspondence away from the low-energy conformal limit, to a duality involving the full, asymptotically Ricci-flat background describing three-branes on the conifold. After a discussion of the nature of this duality at the string theory level (prior to taking any limits), we concentrate on the intermediate-energy regime where excited string modes are negligible but the branes are still coupled to the bulk. Building upon previous work, we are able to characterize the effective D3-brane worldvolume action in this regime as an IR deformation of the Klebanov-Witten $\mathcal{N}=1$ superconformal gauge theory by a specific dimension-eight operator. In addition, we compute the two-point functions of the operators dual to all partial waves of the dilaton on the conifold-three-brane background, and subject them to various checks.

Keywords: Superstrings and Heterotic Strings, D-branes, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The existence of two alternative descriptions of D-brane physics has been clear ever since Polchinski [1] identified the localized hyperplanes where open strings can end [2] with the R-R charged black brane solutions of supergravity [3]. It seems fair to say, however, that, to date, the precise relation between these two approaches has not been completely elucidated. In the early days of the black hole entropy calculations [4], the understanding was essentially that the two descriptions were valid in mutually exclusive regimes; extrapolation from one description to the other would then be warranted only for protected quantities. At the same time, the direct comparison of various quantities in the two pictures, most notably by Klebanov and collaborators [5]–[8], supported the idea that these two perspectives can operate concurrently, in which case one would be dealing with a *duality* at the string theory level.

By adopting this second point of view, and considering a low-energy decoupling limit, Maldacena was able to derive (albeit heuristically) his celebrated correspondence [9]–[11]. Given the impressive body of evidence that has accumulated in support of this gauge/gravity duality [12], one is compelled to take the starting point of Maldacena's argument seriously. It then becomes natural to inquire about the precise nature and origin of the duality that operates at the level of the full-fledged string theory, *before* taking any low energy limits. A related, but more modest, goal is to study this correspondence in a regime of energies that are low enough for the massive string modes to be negligible (thereby cutting the problem down to a more manageable, essentially field-theoretic, size), but not low enough for the branes to decouple from the bulk (which means one is still working away from the Maldacena limit).

The possibility of generalizing in this manner the standard $\mathcal{N}=4$ AdS_5/SYM_4 correspondence away from the conformal limit has been pursued in various works. The authors of [13, 14] argued that the duality in question would equate supergravity on the full asymptotically-flat three-brane background to the effective theory describing the low-energy worldvolume dynamics of D3-branes at strong 't Hooft coupling. Gubser and Hashimoto [15] and Intriligator [16] then characterized the latter theory as an IR deformation of the $\mathcal{N}=4$ fixed point by a specific dimension-eight operator (see also [17]).

To directly examine this non-renormalizable gauge theory would clearly constitute a difficult challenge. The authors of [18] suggested a different line of attack: employing the conjectural duality to extract information about the dual theory directly from the three-brane background.[1] To this end, a recipe to compute correlation functions was developed, and shown to satisfy several non-trivial consistency checks.[2] An element that played an important role in the analysis of [18] is the observation that on the 'worldvolume' side of the duality one also needs to retain the massless closed string modes living in the bulk of ten-dimensional flat space. Subsequent tests of this conjectural intermediate-energy duality have been carried out in [24, 25]. A different approach to duality in the full asymptotically-flat three-brane background has been pursued in [26] (see also [27, 28]).[3]

The main purpose of the present paper is to show how this entire story can be generalized to an $\mathcal{N}=1$ setting. Specifically, we will examine the case where the D3-branes live not in Minkowski space, but on the conifold [30], and explore the possibility of elevating the Klebanov-Witten duality [31] away from the conformal limit. We will start out by elaborating on the general form of the duality at the level of string theory in section 2.1. This will be followed up in section 2.2 with a review of the duality in the intermediate-energy regime that is our main focus. After that, we will explain in section 3 how the arguments of [15, 16] can be carried over to the conifold setting, allowing us to identify the specific deformation of the Klebanov-Witten gauge theory that is relevant for the duality in question. Section 4 describes the way in which the dilaton field couples to the D-branes, and in particular reviews the form of the operators in the worldvolume theory that are dual to the dilaton partial waves. Finally, in section 5 we employ the prescription of [18, 22] to work out the two-point correlators of these operators. A summary of our conclusions is given in section 6.

2. Dual Descriptions of D3-branes

2.1 Worldvolume/geometry string duality

Consider a collection of a large number, N, of D3-branes in Type IIB string theory, placed at radial position $r=0$ in a ten-dimensional space $M^{3,1}\times Y$ with metric

$$ds^2=\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)+\left(dr^2+r^2ds_X^2\right), \tag{2.1}$$

[1] This approach is similar in spirit to that of [19]–[21].

[2] Interestingly, an identical recipe was derived simultaneously in [22], in the context of the supergravity background dual to NCYM [23, 21].

[3] Although not directly related to these developments, for completeness we would also like to draw attention to the recent proposal [29] for a holographic description of flat space.

where the coordinates inside the first (second) parentheses denote directions parallel (transverse) to the branes. In order for the six-dimensional transverse manifold Y to be Ricci-flat, we demand that its five-dimensional base X be Einstein, with positive curvature. The discussion in this section will apply to any such X; but the specific examples we have in mind are $X = \mathbf{S}^5$, in which case Y is simply $\mathbb{R}^6$ (relevant to the discussion in [6, 13, 15, 16, 18, 24, 25]), and $X = T^{11} \equiv [\mathrm{SU}(2) \times \mathrm{SU}(2)]/\,\mathrm{U}(1)$, in which case Y is the conifold [30, 31] (relevant to the analysis in the remaining sections of this paper). Notice that the point $r = 0$ is singular in all cases except $X = \mathbf{S}^5$.

The physics of this D3-brane system can be described from two quite distinct perspectives. On the one hand, we can adopt the above 'worldvolume' perspective, and consider the branes as hyperplanes with intrinsic dynamics, localized at $r = 0$. The excitations of the system are then of two kinds: open strings with endpoints anchored on the branes, and closed strings moving about in the ten-dimensional metric (2.1). On the other hand, we can subscribe to a 'geometry' perspective, replacing the explicit hyperplanes with the solitonic solution of the string equations of motion that carries the same charge and mass. To leading order in α' this is of course the black three-brane solution,[4] with metric [3, 33]

$$\begin{aligned} ds^2 &= H^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H^{1/2}(dr^2 + r^2 ds_X^2)\,, \\ H(r) &= 1 + \frac{R^4}{r^4}\,, \qquad R^4 = \frac{4\pi^4 N g_s l_s^4}{\mathrm{Vol}(X)}\,, \end{aligned} \tag{2.2}$$

a constant dilaton $e^{\bar{\phi}} = g_s$, and N units of Ramond-Ramond flux through X. The above metric describes a geometry with an asymptotically flat region $r > R$, and a throat extending from $r \sim R$ down to a horizon at $r = 0$. On this background, closed strings are the only allowed excitations.

As explained in the Introduction, following Maldacena's argument [9] back to its starting point, we interpret the success of the AdS/CFT correspondence as (circumstantial) evidence for a *duality* between the above two perspectives. To try to make this 'worldvolume/geometry' correspondence more precise, let us focus attention on a specific physical process: closed string scattering off the D3-branes. The concrete claim is then that we can obtain the same result for the scattering amplitude in two different ways:

(i) Summing over worldsheets with an arbitrary number of holes and a fixed number h of handles, with the Ricci-flat background metric (2.1).

(ii) Computing on a single worldsheet with h handles and no holes, with the black three-brane background (2.2).

For the lowest t-channel poles, and for $h = 0$, the agreement between these two calculations was first verified to leading order in the number of holes in [34]–[36], and was then argued to hold to next-to-leading order in [37]. The expectation for the agreement to extend to all orders is ultimately a reflection of open/closed string duality [38]–[41].[5]

[4] See [32] for a recent analysis of a particular next-to-leading-order correction to this background.

[5] Notice that if we work at unrestricted energies, then to reproduce the *massive* t-channel poles/cuts

In description (i), the worldvolume picture, one naturally wonders where the curved geometry is hidden. The answer is that it is implicit in the sum over holes: each boundary gives rise to a tadpole for the graviton, as well as for the other closed string modes. The claim is then that summing over these tadpoles will effectively reproduce the non-trivial background.

Conversely, in description (ii), the geometry picture, one appears to be missing the open string modes, which are known from the worldvolume picture to transform in the adjoint of the $\mathrm{U}(N) \simeq \mathrm{U}(1) \times \mathrm{SU}(N)$ gauge group. To be more precise, what appears to be missing is the $\mathrm{SU}(N)$ part: the $\mathrm{U}(1)$ degrees of freedom describe the center-of-mass motion of the D-brane stack, and must therefore correspond to the zero-mode fluctuations about the black three-brane background (2.2), i.e., the Goldstone modes associated with the symmetries broken by the branes (see, e.g., [42]). The remarkable lesson of the AdS/CFT correspondence is that the $\mathrm{SU}(N)$ part is in fact also present in description (ii), encoded in the closed string degrees of freedom that live in the near-horizon region of the black three-brane background.

Notice that this last point implies that the specific scattering process we are considering is more generic than one might have thought at first sight, because through a judicious choice of the ingoing and outgoing closed string states we can selectively excite the various degrees of freedom intrinsic to the branes. Indeed, this use of closed string scattering as a probe of brane structure lies at the heart of the GKPW prescription [10, 11] for computing correlation functions in the standard AdS/CFT setting, and is also the organizing principle behind the recipe formulated in [18].

Of course, the main obstacle on the way to making the above worldvolume/geometry duality more explicit is the different regimes of validity of descriptions (i) and (ii). We know that for $g_s N < 1$ the sum in (i) is well-defined as an asymptotic expansion.[6] In this regime we can also carry out the computation in (ii) (where we have $R < l_s$), as long as we restrict ourselves to the $r > R$ region (i.e., we only consider processes in which the momentum transferred to the D-branes is small in $1/R$ units). But even without this restriction, notice that the difficulty here is simply that one cannot in general carry out a *perturbative* expansion of the non-linear sigma model in powers of α', because $R < l_s$.[7] This difficulty notwithstanding, the important point for the purpose of formulating the duality is that sensible meaning can still be ascribed to description (ii), in terms of the strongly-coupled two-dimensional theory.[8]

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that are obtained in (i), we would additionally need to consider in (ii) non-trivial backgrounds for the corresponding closed string modes. These backgrounds would decay exponentially with radial distance, with a decay constant of order the string length, and so would only be relevant very close to $r = 0$.

[6]For the discussion of regimes of validity in this paragraph and the next, in the spirit of [43] we are roughly dividing the $g_s N$ half-axis into two regions, $g_s N < 1$ and $g_s N > 1$, but of course, strictly speaking we need to exclude the intermediate region $g_s N \sim 1$ (and, evidently, the corresponding descriptions are better justified for $g_s N \ll 1$ or $g_s N \gg 1$, respectively).

[7]Of course, there is the additional complication of how to treat the background RR field; but this is clearly just a technical issue. For progress on this question, see, e.g, [44, 37, 45].

[8]The latter might even admit a more explicit treatment in terms of a ‘string bit’ formalism, as in [46, 47] (see also [48]–[50] and references therein).

In the opposite direction, when $g_sN > 1$ we know description (ii) is well-defined; but the *perturbative* sum in (i) is not. Nonetheless, assuming that there exists a non-perturbative definition of string theory, we can regard the sum over holes as a metonym for the corresponding computation in the strongly-coupled string theory. Of course, the key question here is whether this computation can still be meaningfully formulated in an open string language. A non-perturbative formalism that could perhaps be well-suited for this purpose is open string field theory [51]. Incidentally, note that in this latter context it has been argued that closed strings can be directly expressed in terms of the open string fields (see, e.g., [52]–[54]), and so do not need to be added as independent degrees of freedom. If correct, this would allow a 'purely four-dimensional' formulation of the worldvolume side of the duality, which would be more in line with what we have gotten used to in the AdS/CFT setting. For related work, see [55]–[58].

The other piece of string theory lore that might appear to be in conflict with the worldvolume/geometry duality assumed in [9] and advocated here is the Fischler-Susskind mechanism [59]–[62]. The original application of this mechanism to worldsheets with holes dates back to the pre-D-brane era, and was consequently carried out in the context of a space-filling Dp-brane (i.e., $p = 25, 9$ for the bosonic and super- string cases, respectively). The issue was that holes were found to give rise to divergences, and to cancel these it was necessary to modify the *closed* string background [61, 62].[9]

In more detail: the disk, for instance, can be represented as a sphere with a hole cut out, where one integrates over the radius and position of the hole.[10] A divergence arises from the lower limit of the integral over the size of the hole, where the surface is closing up into a sphere. In this limit, the leading contribution to the amplitude can be seen to factorize into three pieces: the sphere with an additional graviton/dilaton vertex operator, a graviton/dilaton propagator, and a disk with the respective graviton/dilaton vertex as its only insertion. This last factor is a tadpole amplitude, expressing the possibility for the space-filling brane to emit (or absorb) a graviton/dilaton. Since the source is position-independent, the massless particle is emitted with zero momentum, which means that its propagator must be evaluated on-shell. This is then the physical origin of the divergence.

The insight of Fischler and Susskind is that the theory can be 'renormalized' by shifting the background metric/dilaton, which to leading order amounts to inserting a single graviton/dilaton vertex on the sphere, in such a way as to cancel the divergence. The physical interpretation of this is that, in order to have a well-defined perturbative expansion, one is forced to perturb about the vacuum that solves the 'loop'-corrected equations of motion. Shifting to this vacuum in effect amounts to summing up the tadpoles, to the order one is working at. In the D-brane case, then, the shift should take us to the corresponding black

[9] As is well-known, in the case of the unoriented open string, and for a specific choice of the gauge group, it is possible to cancel the annulus divergence against similar divergences on the Möbius strip and the Klein bottle (or, in modern terms, to adjust the number of D-branes so as to cancel the charge of the orientifold plane).

[10] These additional moduli compensate for the fact that the sphere has a larger conformal Killing group than the disk.

brane background. Indeed, it is precisely through the Fischler-Susskind mechanism that open string backgrounds are able to contribute to the Einstein equation (i.e., to the beta functional for the metric) [61, 62].

In a first reading, the above story might seem to imply that one must necessarily consider explicit D-branes (holes on the worldsheet) *and* the corresponding black brane geometry (closed string vertex insertions) *at the same time*, therefore invalidating the duality we have been pursuing. But assume for a moment that this is correct, and that at the string theory level there is only one description, involving explicit D-branes in the background they themselves generate. The problem is then that, starting from this *unique* description and taking a *single* low-energy limit, one cannot possibly obtain the two sides of the AdS/CFT correspondence.[11]

The observation that shows us the way out of this seeming paradox is that, in the case of *localized* (as opposed to space-filling) D-branes, the kinematics no longer forces us to work right on top of the massless particle pole, a point that has been stressed in [66] at the level of the disk. This means that for D3-branes, in particular, the individual amplitudes in the sum (i) contain no divergence associated with massless closed string tadpoles,[12] and there is consequently no need to invoke Fischler and Susskind.

The physical point here can be better appreciated by referring back to a field-theoretic analog, considering a massless scalar field $\varphi(x)$ with a linear external source $\lambda J(x)$ (see, e.g., [70]). In such setting one has the option of computing physical quantities *either* by starting in the $\langle\varphi(x)\rangle = 0$ vacuum and summing up the J-tadpoles to all orders in λ, *or* by shifting from the beginning to the $\langle\varphi(x)\rangle = \varphi_J(x)$ vacuum, where $\varphi_J(x)$ is the solution to the φ equation of motion with the source included. The agreement between these two methods is of course a triviality in the field theory context, but its analog in the string setting is precisely the surprising equivalence (i)=(ii), which contains in particular the remarkable AdS/CFT correspondence.

There are two simple points one can make using this analogy. One is that the expansion about $\langle\varphi(x)\rangle = 0$ is well-defined except in the case where the source $J(x)$ is position-independent, which would be precisely the analog of the space-filling D-brane. The other is that, if one carries out the sum over tadpoles only to finite order in λ, then one has in effect shifted to a vacuum that no longer satisfies $\langle\varphi(x)\rangle = 0$, but is not yet the one where $\langle\varphi(x)\rangle = \varphi_J(x)$, a situation which is analogous to having to discuss explicit holes and curved geometry at the same time. Based on this analogy, then, the Fischler-Susskind mechanism — or more accurately, its generalization to the case with no divergences (perhaps along the

[11] To try to avoid this problem, it has been argued in [63] (see also [64]) that the duality at the level of string theory has on its geometry side the black three-brane background, and on its worldvolume side, explicit D3-branes living on the *same* background. We find it difficult to make sense of a duality statement of this kind, because there is clearly more on one side than on the other. An additional problem is that, starting with this purported equivalence, the decoupling limit does not yield the correct AdS/CFT statement: there is now a redshift factor on both sides of the duality, and so one retains the full open string spectrum instead of just the massless modes. Related work by the same author may be found in [65].

[12] Starting at the annulus level, worldsheets with holes do give rise to divergences, but these are *open* string effects associated with D-brane recoil [67, 68]. Their counterparts in description (ii) are the usual zero-mode divergences present in the perturbation expansion about a soliton background (see, e.g., [69]).

JHEP06(2003)049

lines of [66]) — is then understood to be precisely the device that perturbatively implements the shift from description (i) to description (ii). For other perspectives on this question, see [37, 39]. It would clearly be desirable to try to work this out in a more explicit manner, and we hope to return to this set of issues elsewhere. In the remainder of the present paper, however, we will concentrate on analyzing this worldvolume/geometry duality in a simplified (essentially field-theoretic) setting.

2.2 The duality at low and intermediate energies

Having discussed the duality at the level of string theory, let us now review the different ways in which one can reduce it to more manageable forms. From this point on we work in the $g_s N \gg 1$ regime, where $R \gg l_s$ and one has relatively good control over description (ii), the geometry picture. To attain the desired simplicity, we restrict attention to processes with energies lower than some cutoff Λ that is in turn smaller than the string scale, $\omega < \Lambda < 1/l_s$. On the worldvolume side, this means that we need only consider the massless closed and open string modes; the description is then in terms of the effective low-energy bulk theory (supergravity plus higher-derivative corrections) in the Ricci-flat ten-dimensional space (2.1), coupled to the corresponding effective theory on the D-brane worldvolume. On the geometry side, one similarly retains only the lowest closed string modes away from the branes, but in addition, the presence of the redshift factor seen in (2.2) implies that by moving towards $r = 0$ one can have excitations with larger local proper energies. More precisely, at radial position r the effective cutoff on proper energies is $\Lambda H(r)^{1/4}$, so modes with locally-measured energies of order $1/R$ are present in the region $r \leq r_R$, where (for $\Lambda \leq 1/R$)

$$r_R = \frac{\Lambda R^2}{\left[1-(\Lambda R)^4\right]^{1/4}} \,. \tag{2.3}$$

Similarly, modes with string-scale proper energies live inside $r \leq r_{l_s}$, where

$$r_{l_s} = \frac{\Lambda R l_s}{\left[1-(\Lambda l_s)^4\right]^{1/4}} \,. \tag{2.4}$$

Notice that, for $\Lambda < 1/R$, the survival of modes with proper energies larger than $1/R$ is made possible only by the presence of the branes. We can consequently regard the region $r \leq r_R$ as a rough indication of the portion of the geometry (2.2) that is expected to be dual to the D3-brane worldvolume theory (in a sense that will be made more precise below).

Given the large splitting between the 'Kaluza-Klein'[13] and string energy scales, $1/R \ll 1/l_s$, even in this simplified setting there are (at least) two qualitatively distinct energy regimes available. Maximal simplicity is attained in the low energy range $\Lambda \ll 1/R \ll 1/l_s$, where, to zeroth order in ΛR, the branes decouple from the bulk [6, 7], the D3-brane world-

[13]Note that, strictly speaking, in the general case a Kaluza-Klein interpretation is not possible. That is to say, even though one can of course always expand the ten-dimensional fields in terms of X-harmonics, the coefficients in this expansion cannot be regarded as fields living on the 'remaining' five-dimensional spacetime unless the geometry factorizes, which is the case only in the AdS/CFT limit.

volume theory becomes conformal, and the portion $r \leq r_R$ of the geometry (2.2) reduces to the near-horizon $\mathrm{AdS}_5 \times X$ form. We are then left with Maldacena's extraordinary correspondence [9, 12].

It is customary to formulate the above decoupling limit in units such that $l_s \to 0$ with R/l_s held fixed, in which ω can be kept arbitrary (i.e., *after* taking the limit one can remove $\Lambda \to \infty$). To focus attention on the $r \leq r_R$ region, one then introduces a new radial coordinate[14] $u \equiv r/R^2$, which is kept fixed in the limit (one then has $u_R = \Lambda \to \infty$). The possibility to express the duality in this manner is based on the fact that, having found an IR fixed point by moving along the RG flow associated with the duality, one is of course free to consider the resulting conformal theory at unrestricted energies. In either language, if we additionally choose to work in the limit of infinite 't Hooft coupling, $g_{YM}^2 N \sim g_s N \to \infty$, then $r_{l_s}/r_R \to 0$ ($u_{l_s} \to 0$), and so the description in the geometry picture is purely in terms of the supergravity modes.

Notice that, in this low-energy regime, (2.3) simplifies to

$$r_R = \Lambda R^2 \qquad (u_R = \Lambda)\,, \tag{2.5}$$

which manifestly shows that the physics associated with lower energy scales in the world-volume theory takes place at smaller values of the radial coordinate in the geometry picture. We recognize (2.5) as the statement of the well-known UV-IR connection [71, 72], linearly translating the bulk radial coordinate r into an energy scale in the field theory. This allows us to identify the redshift factor in (2.2) as the physical basis for this connection (as well as its relation to RG flow [73]), and to interpret (2.3) as its generalization to the entire three-brane background. Notice that the nature of this connection is more subtle than it is sometimes assumed: rather than a direct one-to-one correspondence between a given energy scale ω and a fixed radial position r_ω, we find that processes with a given energy ω can take place at all $r \leq r_\omega$.

Alternatively, one can work away from the $\Lambda R = 0$ limit, where the branes no longer decouple from the bulk, the effective D3-brane theory is not conformally invariant, and the region $r \leq r_R$ of (2.2) is not purely $\mathrm{AdS}_5 \times X$ [6] and [13]–[16]. A point that cannot be overemphasized here is that *one should not mistake the lack of brane-bulk decoupling for the absence of a duality* — as explained in the previous subsection, a duality exists even at the string level, prior to any limits.

In this intermediate-energy regime the cutoff can be dialed across the entire sub-stringy range $0 < \Lambda < 1/l_s$, which contains the 'almost-Maldacena' extreme $0 < \Lambda \ll 1/R$, but also includes cases where much higher energies are allowed. From now on we restrict ourselves to $\Lambda \ll 1/l_s$; this suppresses stringy corrections in the bulk effective action and allows us to carry out the analysis within a supergravity framework. Notice that the above restriction translates into $\Lambda R \ll (g_s N)^{1/4}$, which still allows moderately high energies in the sense of $\Lambda \gg 1/R$. This is an important observation because it is the energy scale $1/R$, and

[14] As long as $g_s N$ is held fixed, the rescaling $u \equiv r/R^2$ is morally equivalent to the one originally employed in [9], $u \equiv r/l_s^2$. On the other hand, if we allow ourselves the freedom to change $g_s N$, then the former rescaling is more useful, as we will see momentarily. See also the discussion in [72].

not $1/l_s$, that controls the higher-derivative terms in the D3-brane worldvolume effective action [13]–[15] and [18]. In other words, the theory becomes conformal in the Maldacena limit $\Lambda R \to 0$, but remains non-conformal as long as ΛR is finite, even if we allow ourselves to send $\Lambda l_s \to 0$.

An important difference with the AdS/CFT case is that this time we cannot change to units where ω is completely unrestricted unless we simultaneously confine ourselves to the limit of infinite 't Hooft coupling, in order to be able to send $l_s \to 0$ while holding R fixed. This is precisely the 'double scaling limit' introduced by Klebanov in the seminal paper [6], where the existence of an intermediate-energy duality was first proposed. Klebanov's limit places us at a particularly tractable corner of the full parameter space where the intermediate-energy correspondence discussed in the present subsection is defined. For simplicity we will work mostly in this corner; but even in this case it should be borne in mind that the duality under consideration has been obtained by restricting ourselves to the substringy domain, and therefore comes with the built-in cutoff Λ. This point is crucial in trying to make sense of the duality statement, because the theories one equates in this intermediate-energy domain are inevitably non-renormalizable.

The conclusion is then that, for $0 < \Lambda R \ll (g_s N)^{1/4}$, the effective intermediate-energy theory on the D3-brane worldvolume, *coupled to* supergravity on the Ricci-flat space (2.1), is dual to supergravity on the full three-brane background (2.2), with an energy cutoff Λ enforced on both sides of the correspondence.[15] Since $g_{YM}^2 N \gg 1$, the worldvolume theory is strongly coupled, and so is not given merely by the Born-Infeld action [15]. To have a more explicit duality statement, it is thus necessary to determine the intermediate-energy effective action for a large number of D3-branes at strong 't Hooft coupling — undoubtedly a daunting task. Fortunately, as explained in [15, 16] for the case $X = \mathbf{S}^5$, string-theoretic information highly constrains the possible form of the required action. In section 3 we will explain how the arguments of [15, 16] can be generalized to the case $X = T^{11}$.

We should also say a word about the interactions allowed in this energy regime. Since Newton's constant $\kappa \sim g_s l_s^4$ is IR irrelevant, to zeroth order in Λl_s (i.e., in the strict $g_s N \to \infty$ limit) the supergravity theories in both the worldvolume and the geometry pictures become free.[16] The only remaining interactions then take place on the D-brane worldvolume, with coupling strength $g_s N \to \infty$. On the geometry side, the net effect of these interactions on the supergravity fields should be summarized by the boundary conditions enforced at the horizon $r = 0$. As has been emphasized in [18], even in this limit the branes *do not* strictly decouple from the bulk: the coupling in question is not of order κ or g_s, but of order $R^4 \sim g_s N l_s^4$. Excitations on the D-branes can repeatedly transmute into supergravity modes, wander off into the bulk, and then come back into the worldvolume. If we worked away from the strict $g_s N \to \infty$ limit (to leading order

[15]Roughly speaking, then, the proposal is that the worldvolume theory reproduces the physics of the $r < r_R$ region of the three-brane background (which includes much more than the near-horizon $\mathrm{AdS}_5 \times X$ region), and what remains, on both sides of the correspondence, is simply supergravity on $M^{3,1} \times Y$. Of course, the lack of decoupling between the branes and the bulk makes it difficult to be more precise about this split.

[16]We are grateful to Juan Maldacena for making this point clear to us.

in Λl_s, say), then the supergravity theories would become interacting (albeit extremely weakly coupled), but if desired we could revert to the free supergravity setup by sending $g_s \to 0$.

3. Intermediate-energy worldvolume effective action

As explained in the Introduction, the main purpose of the present paper is to study the worldvolume/geometry duality in the conifold setting, at energies low enough to be able to neglect massive string modes, but still away from the decoupling limit of [9]. Recall from section 2 that on the worldvolume side of the duality we are placing N D3-branes in the ten-dimensional space (2.1), where from now on we set $X = T^{11}$. The latter is a five-dimensional space that can be defined as the quotient of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ by the diagonal U(1) generated by $\sigma_3^L + \sigma_3^R$; topologically it is $S^2 \times S^3$. With this choice Y, the six-dimensional cone over X, is the conifold, a Calabi-Yau manifold with a conical singularity [30]. The stack of D3-branes is placed right at the singularity, $r = 0$.

The worldvolume theory describing this system at extremely low energies has been constructed by Klebanov and Witten [31] (see also [74]). It is a superconformal gauge theory with $\mathcal{N} = 1$ supersymmetry, gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$, and chiral superfields $A_i, B_k,\ \ i,k = 1,2$. A_i transforms as $(\mathbf{N}, \bar{\mathbf{N}})$ and B_k as $(\bar{\mathbf{N}}, \mathbf{N})$ under $\mathrm{SU}(N) \times \mathrm{SU}(N)$. In addition, the theory has a superpotential

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl} A_i B_k A_j B_l\,, \tag{3.1}$$

which as explained in [31] would be non-renormalizable if considered as a perturbation of free field theory, but is in fact exactly marginal when understood as a deformation of the IR fixed point of the $\lambda = 0$ gauge theory.

On the geometry side, we start out with the three-brane background (2.2), with $X = T^{11}$. The near-horizon limit of this spacetime is $\mathrm{AdS}_5 \times T^{11}$, and so, following the arguments of [9], one concludes that Type IIB string theory propagating on this near-horizon geometry is dual to the above superconformal gauge theory [31] (for a review of this and related dualities, see [75, 76]). The $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ part of the isometry group of T^{11} acts on the chiral superfields in a simple way; one SU(2) acts on the A_i and the other on the B_k. The remaining part of the T^{11} isometry group, $\mathrm{U}(1)_R$, plays the role of R-symmetry of the $\mathcal{N} = 1$ gauge theory. The four matter fields A_i, B_k carry an R-symmetry charge $s = 1/2$.

Let us now examine the duality at intermediate energies, away from the $\Lambda R = 0$ limit (where Λ is the UV cutoff introduced in section 2.2). More concretely, out of the various regimes discussed in section 2.2, we will henceforth focus attention on the Klebanov limit, $l_s \to 0, g_s N \to \infty$, with $R \sim (g_s N)^{1/4} l_s$ held fixed [6]. On the geometry side this implies that we will be working with free supergravity on the full asymptotically-Ricci-flat background (2.2). On the worldvolume side, we are left with free supergravity propagating on the Ricci-flat spacetime (2.1), *coupled to* the intermediate-energy effective lagrangian $\mathcal{L}_{\mathrm{D3}}$ on the D3-brane worldvolume, a non-renormalizable theory whose form we wish to determine. At low energies $\mathcal{L}_{\rm D3}$ must reduce to the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ superconformal gauge theory described above (mirroring the fact that the full geometry (2.2) reduces to $\mathrm{AdS}_5\times T^{11}$ as $r\to 0$), and so the theory we are after can be characterized as an IR deformation of the Klebanov-Witten theory [31]. We can thus write, without loss of generality,

$$\mathcal{L}_{\rm D3} = \mathcal{L}_{\rm SCFT} + \sum h_\Delta R^{\Delta-4}\mathcal{O}_\Delta\,, \tag{3.2}$$

where the sum runs over non-renormalizable (IR irrelevant) gauge-invariant operators $\mathcal{O}_\Delta$ with dimension $\Delta > 4$, and h_Δ is a dimensionless coupling.

Now, a crucial point is that the operators $\{\mathcal{O}_\Delta\}$ appearing in (3.2) must preserve the same symmetries as the three-brane background (2.2). This means that we are only interested in IR irrelevant operators that are Lorentz scalars, $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_R$ invariant, and $\mathcal{N}=1$ supersymmetric. In the $X=\mathbf{S}^5$ case, two independent arguments made it possible to zero in on the desired operators [15, 16]. As we will now see, both of these can be carried over to our $X=T^{11}$ case.

The first of these arguments is based on the well-known fact that, in the AdS/CFT setting, a Lorentz- and gauge-invariant operator of conformal dimension Δ is mapped in the geometry side into a scalar mode of mass m, with Δ and m related through [10, 11]

$$\Delta = 2+\sqrt{4+m^2R^2}\,. \tag{3.3}$$

This implies that, in the strong-'t Hooft-coupling regime that is of interest to us, operators dual to excited string modes (with masses $m\sim 1/l_s$) acquire large anomalous dimensions $\Delta\sim (g_sN)^{1/4}\to\infty$. Following [15], we can consequently restrict the sum in (3.2) to run only over operators dual to supergravity modes. Analyzing the full mass spectrum of Type IIB supergravity on $\mathrm{AdS}_5\times T^{11}$, which has been derived in [77, 78] using harmonic expansion on T^{11} (see also [33, 79]), we find that the only mode compatible with the required symmetries is dual to the *non-chiral* superfield $Q^0\equiv \mathrm{Tr}(W^2e^V\bar{W}^2e^{-V})$, where W_α and V are respectively the field strength (chiral) and gauge field (vector) superfields. Q^0 contains the descendant $\mathrm{Tr}(F_1^4+F_2^4)$, with $\Delta=8$. Since this operator belongs to a long multiplet, in principle it is not expected to have protected dimension; but the Klebanov-Witten duality predicts otherwise. The existence of such *a priori* unprotected multiplets with rational conformal dimensions is a peculiarity of this $\mathcal{N}=1$ AdS/CFT correspondence [33, 77, 78].

Applying the logic of [15] to our case we are thus led to conjecture that the D3-brane worldvolume theory in the intermediate-energy regime of interest to us is

$$\mathcal{L}_{\rm D3} = \mathcal{L}_{\rm SCFT} + h_4R^4\mathcal{O}_8\,, \tag{3.4}$$

where

$$\mathcal{O}_8\equiv \mathrm{Tr}(F_1^4+F_2^4)+\cdots\,, \tag{3.5}$$

the dots denoting the $\mathcal{N}=1$ supersymmetric completion. In principle, $\mathcal{O}_8$ could also include an admixture of a double-trace operator of the type $\mathrm{Tr}(F^2)\,\mathrm{Tr}(F^2)$. Notice, however, that $\mathcal{O}_8$ should not be primarily double-trace, for such a deformation has been argued to be dual to a non-local string theory [80]–[83], which is certainly not what we have on our geometry side.

Following [16], a second argument can be given to show that the deformation in (3.2) must involve a single operator of dimension eight, precisely as we have concluded in (3.4). Indeed, part of the considerations of [16] are based just on scaling properties of the metric, and can be readily applied to any background of the form (2.2). As explained in [16], when one considers a background of this type, with $H(r) = h + R^4/r^4$, a scaling argument shows that the absorption probability depends on h only through the effective coupling $h_{eff} \sim hR^4/x^4$. This means that, on the worldvolume side, the unique deformation parameter h must multiply an operator in the worldvolume lagrangian whose dimension is exactly $\Delta = 8$.[17]

To summarize, with the help of [15, 16] we have been able to postulate the specific form (3.4) for the D3-brane worldvolume intermediate-energy lagrangian relevant to the worldvolume/geometry duality in the Klebanov limit, $l_s \to 0$ with R fixed. As was emphasized in [18] and elaborated on in section 2.2 of the present paper, D-brane physics implies that $\mathcal{L}_{\text{D3}}$ *cannot* by itself be the complete theory dual to supergravity on the full three-brane background (2.2): since we are working away from the Maldacena limit, the worldvolume theory remains coupled to (in our case free) supergravity in the Ricci-flat bulk (2.1).[18] To linear order in the supergravity fields, the relevant brane-bulk interaction lagrangian simply couples each supergravity mode to its dual gauge-invariant operator. In the following section we will discuss this coupling for the specific case of the partial waves of the dilaton, which will then enable us in section 5 to compute the two-point functions of the corresponding operators.

4. Coupling of the dilaton to the D-branes

To understand how the dilaton field couples to D3-branes placed at a conifold singularity, it helps to review this first for the flat-space ($X = \mathbf{S}^5$) case relevant to [6, 13, 15, 18]. Letting x^μ and y^i denote cartesian coordinates respectively parallel and transverse to the stack of D3-branes, we know that the interaction is of the form

$$S_{\text{int}} = \int d^4x \sum_l R^{2l} \, \partial_{i_1 \ldots i_l} \phi(x; y)\big|_{y=0} \mathcal{O}^{i_1 \ldots i_l}(x) \,, \tag{4.1}$$

with $\mathcal{O}^{i_1 \ldots i_l}(x)$ the gauge theory operators determined explicitly in [84]. The appearance in (4.1) of derivatives with respect to the cartesian coordinates y^i correlates with the fact that these operators have the schematic form $\mathcal{O}^{i_1 \ldots i_l}(x) \sim \text{Tr}\left[(F^2 + \cdots)\Phi^{i_1} \cdots \Phi^{i_l}\right]$, where $\Phi^i = X^i/R^2$ are the scalar fields associated with the transverse position of the branes.

Both in the worldvolume and in the geometry sides of the duality it is natural to decompose the dilaton field into partial waves,

$$\phi(x; \vec{y}) = \phi(x; r, \hat{y}) = \sum_l \phi^{(l)}_{\mathbf{m}}(x; r) Y_{l\mathbf{m}}(\hat{y}) \,, \tag{4.2}$$

[17] To avoid misunderstandings we should remark that most of the other deductions of [16] rely on the existence of sixteen supercharges, and thus cannot be directly carried over to our $\mathcal{N} = 1$ setting.

[18] Of course, if desired one could in principle integrate out the supergravity degrees of freedom. Notice, however, that this would give rise to non-local interactions on the worldvolume.

where $Y_{l\mathbf{m}}(\hat{y}) \equiv C_{l\mathbf{m}}^{i_1 \ldots i_l} \hat{y}^{i_1} \cdots \hat{y}^{i_l}$ is one of the $(l+3)(l+2)^2(l+1)/12$ $\mathbf{S}^5$-harmonics corresponding to the lth partial wave [85], and $\hat{y} \equiv \vec{y}/r$ is a unit vector. It is important therefore to understand how the partial waves $\phi^{(l)}$ are related to the differentiated dilaton appearing in (4.1). Comparing the Taylor expansion of $\phi(x;\vec{y})$ about $\vec{y}=0$ against the right-hand-side of (4.2), it is easy to deduce that [84]

$$\begin{aligned}
\phi^{(0)}(x;r) &\propto \left[\phi(x;\vec{y}) + \frac{1}{12} r^2 \partial_i \partial_i \phi(x;\vec{y}) + \cdots \right]_{\vec{y}=0} , \\
\phi^{(1)}_i(x;r) &\propto r \left[\partial_i \phi(x;\vec{y}) + \frac{1}{16} r^2 \partial_i \partial_j \partial_j \phi(x;\vec{y}) + \cdots \right]_{\vec{y}=0} , \\
\phi^{(2)}_{ij}(x;r) &\propto r^2 \left[\partial_i \partial_j \phi(x;\vec{y}) + \cdots \right]_{\vec{y}=0} ,
\end{aligned} \tag{4.3}$$

and so on (the dots represent terms with more derivatives).

Now, recall from section 2 that the explicit stack of D3-branes lives by definition on the worldvolume side of the duality, where the spacetime metric is (2.1). For the flat space case, and for a given partial wave, the solution to the radial dilaton equation of motion that is regular at $\vec{y}=0$ is $\phi^{(l)}(r) \propto r^{-2} J_{l+2}(qr)$, where

$$q^2 \equiv -k^\mu k_\mu , \tag{4.4}$$

with k_μ ($\mu = 0,1,2,3$) the momentum along the D3-brane worldvolume directions. The above solution vanishes like r^l as $r \to 0$ (where the D-branes are located), and so the gauge theory operators must couple not to $\phi^{(l)}(r)$ but to $r^{-l}\phi^{(l)}(r)$, which is finite at the location of the branes. And indeed, using (4.3) we see that this is what (4.1) is telling us. We can consequently rewrite the coupling in the form

$$S_{\mathrm{int}} = \int d^4x \sum_{l,\mathbf{m}} \left[\left(\frac{R^2}{r}\right)^l \phi^{(l)}_{\mathbf{m}}(x;r) \right]_{r=0} \mathcal{O}^{l\mathbf{m}}(x) , \tag{4.5}$$

which shows the explicit pairing between each of the components of a given dilaton partial wave and the gauge theory operator with the same SU(4) quantum numbers.

Let us now move on to the conifold. For the relevant $\mathcal{N}=1$ theory [31], the superfields A_i, B_k (or more precisely, their scalar component fields a_i, b_k) play a role analogous to the scalar fields Φ^i of the $\mathcal{N}=4$ case, so it is convenient to use the corresponding coordinates on the conifold. To set our notation, let us first review how these coordinates are introduced [30, 31]. Start with the equation that defines the conifold,

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 , \tag{4.6}$$

and define a radial variable r through

$$r^3 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 . \tag{4.7}$$

Letting

$$z_1 = z_1' + i z_2' , \qquad z_2 = z_2' + i z_1' , \qquad z_3 = z_3' - i z_4' , \qquad z_4 = z_4' - i z_3' , \tag{4.8}$$

we can rewrite (4.6) and (4.7) in the form [31]

$$z'_1 z'_2 - z'_3 z'_4 = 0 \tag{4.9}$$

and

$$r^3 = |z'_1|^2 + |z'_2|^2 + |z'_3|^2 + |z'_4|^2 \,. \tag{4.10}$$

Next, introduce coordinates a_i, b_k $(i, k = 1, 2)$ that solve (4.9), through

$$z'_1 = a_1 b_1 \,, \qquad z'_2 = a_2 b_2 \,, \qquad z'_3 = a_1 b_2 \,, \qquad z'_4 = a_2 b_1 \,, \tag{4.11}$$

in terms of which

$$r^3 = (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2) \,. \tag{4.12}$$

The rescaling $a_i \to \lambda a_i$, $b_k \to \lambda^{-1} b_k$ leaves (4.11) invariant, and we can use this freedom to arbitrarily set

$$|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2 \equiv \mathcal{R}^2 \,. \tag{4.13}$$

This describes two three-spheres of radius $\mathcal{R} = r^{3/4}$ embedded in $\mathbb{R}^4$, so T^{11} (i.e., the conifold at a fixed $r > 0$) is $\mathbf{S}^3 \times \mathbf{S}^3$ modulo the freedom to shift the phase of a_i, b_k, which was not eliminated by the condition (4.13).[19]

We can again expand the dilaton field into harmonics [77, 78],

$$\phi(x; r, \widehat{a}, \widehat{b}) = \sum_{\mathbf{lm}} \phi^{(1)}_{\mathbf{m}}(x; r) Y^{\mathbf{lm}}_{(0)}(\widehat{a}, \widehat{b}) \,, \tag{4.14}$$

where $\mathbf{l} \equiv (l, j, s)$ label a given representation of the T^{11} isometry group, $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$, and $\mathbf{m} \equiv (l_3, j_3)$ identify a specific component of the $(2l + 1)(2j + 1)$-dimensional representation. The harmonics $Y^{\mathbf{lm}}_{(0)}$ are eigenfunctions of the scalar laplacian on T^{11},

$$\Box Y^{\mathbf{lm}}_{(0)}(\widehat{a}, \widehat{b}) = -H_0(\mathbf{l}) Y^{\mathbf{lm}}_{(0)}(\widehat{a}, \widehat{b}) \,, \tag{4.15}$$

with eigenvalues [33, 77, 78]

$$H_0(l, j, s) = 6 \left[l(l + 1) + j(j + 1) - \frac{s^2}{8} \right] . \tag{4.16}$$

The $\mathrm{SU}(2) \times \mathrm{SU}(2)$ quantum numbers (l, j) must be both integers or both half-integers, and the R-symmetry charge s satisfies[20] $|s| \le 2l$, $|s| \le 2j$.

As we will see in the next section, the solution to the radial dilaton equation of motion that is regular at $r = 0$ (where the D3-branes are located) is $\phi^{(1)} \propto r^{-2} J_v(qr)$, where

$$v(l, j, s) = \sqrt{H_0(l, j, s) + 4} \,. \tag{4.17}$$

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[19] As explained in [31], when determining the moduli space of the gauge theory one encounters an equivalent pair of restrictions, and the moduli space is then seen to be (the product of N copies of) the conifold.

[20] In general the harmonics carry one more label, $q \in \mathbb{Z}$, which refers to their charge under the $U_H(1)$ group that defines T^{11} as a coset space (and is not to be confused with (4.4)). For the harmonic $Y^{(l,j,s)}_{(q)}$, the restriction over s reads $|q + s| \le 2l$, $|q - s| \le 2j$, $2l_3 = q + s$, $2j_3 = q - s$. For scalar harmonics $q = 0$, and this is the only case we will consider in the present work.

For $r \to 0$, then, we have $\phi^{(\mathbf{l})} \sim r^{\upsilon-2}$, from which we conclude that the coupling between the branes and the bulk dilaton field should take the form

$$S_{\text{int}} = \int d^4x \sum_{\mathbf{l},\mathbf{m}} \left[\left(\frac{R^2}{r}\right)^{\upsilon-2} \phi^{(\mathbf{l})}_{\mathbf{m}}(x;r)\right]_{r=0} \mathcal{O}^{\mathbf{lm}}(x)\,. \tag{4.18}$$

This is then the conifold analog of the flat-space interaction (4.5). By construction, the above products of powers of r and partial waves $\phi^{(\mathbf{l})}$ are finite at the location of the branes, so this is the only form of the coupling that could make sense.[21] The factor of R^2 has been inserted in order for S_{int} to be dimensionless, knowing that $\mathcal{O}^{\mathbf{lm}}$ has dimension $\Delta = 2+\upsilon$.[22]

The operators $\mathcal{O}^{\mathbf{lm}}(x)$ lying in multiplets with protected dimensions have been identified in [77, 78]. Two types of multiplet-shortening conditions are relevant in this context. The dilaton partial waves with $l = j = s/2$ lie in short multiplets and are dual to descendant operators of the type $\mathrm{Tr}[F_1^2(ab)^s + F_2^2(ba)^s + \cdots]$ contained in the chiral superfields $\Phi^s \equiv \mathrm{Tr}[W_1^2(AB)^s + W_2^2(BA)^s]$, where $W_{1\alpha}$ ($W_{2\alpha}$) is the field strength superfield for the first (second) factor of the $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge group, and one symmetrizes separately over the indices of the A_i and the B_k. (Waves with $l = j = -s/2$ are then related to the corresponding antichiral superfields.) The partial waves with $j = l-1 = s/2$ are in 'semi-long' multiplets, and are dual to descendant operators contained in the 'semi-conserved' superfields $I^s \equiv \mathrm{Tr}[Ae^{V_2}\bar{A}e^{-V_1}W_1^2(AB)^s]$. (Similarly, waves with $l = j-1 = s/2$ are related to $\mathrm{Tr}[Be^{V_1}\bar{B}e^{-V_2}W_2^2(BA)^s]$.)

All of the remaining partial waves lie in long multiplets and couple to operators whose dimension at the conformal fixed point, $\Delta = 2+\upsilon$, is irrational. These are linear combinations of descendants of the various gauge-invariant superfields that can be constructed with the basic building blocks $W^2(AB)^k$, $Ae^V\bar{A}e^{-V}$, $Be^V\bar{B}e^{-V}$, and $W_\alpha e^V \bar{W}_{\dot\alpha} e^{-V}$. Generally one expects mixing between all of the operators carrying a given set of quantum numbers. Notice also that in our setting all of the above operators would receive additional corrections (e.g., of the type $\mathrm{Tr}[F_1^4(ab)^n + F_2^4(ba)^n]$) due to our departure from the conformal limit.

5. Correlation functions

In this section we will calculate the two-point functions of the operators dual to all dilaton partial waves in a conifold background. In addition to providing information about the departure of the gauge theory from the conformal fixed point, this will allow us to test the intermediate-energy prescription of [18] in a background that is non-spherical and non-maximally-supersymmetric.

[21] The appearance of explicit factors of r in (4.5) and (4.18) is quite analogous to what happens in the standard GKPW recipe [10, 11] for, say, AdS_5: in that case the equation of motion for a scalar field φ with mass m has two independent solutions, behaving like $z^{4-\Delta}$ and z^{Δ} for small z, where $z = R^2/r$ (so that $z = 0$ is the boundary of AdS) and Δ is given by (3.3). The corresponding gauge theory operator $\mathcal{O}$ (of dimension Δ) then couples not to the boundary value of $\varphi(z)$, but to the boundary value of $z^{\Delta-4}\varphi(z)$, which is finite at $z = 0$.

[22] For the time being we are taking ϕ to be conventionally normalized (with a factor of $1/2\kappa^2$ in front of its kinetic term), and so dimensionless. In the next section we will work instead with a *canonically* normalized dilaton. To retain (4.18) as it stands, we will then need to absorb a factor of R^4 into $\mathcal{O}^{\mathbf{lm}}$.

The basic idea behind the recipe of [18] for computing correlation functions is to compare the amplitude for dilaton propagation in the geometry and worldvolume pictures. The corresponding dilaton propagators, G and G_0, are defined as solutions to the ten-dimensional linearized dilaton equation of motion,

$$\partial_M \left[\sqrt{-g} g^{MN} \partial_N G\right] = i\delta^{(10)}\left(x^M - x'^M\right), \tag{5.1}$$

in the respective backgrounds (2.2) and (2.1), with $X = T^{11}$. The ten-dimensional coordinates x^M ($M = 0, \ldots, 9$) consist of the directions x^μ ($\mu = 0, \ldots, 3$) parallel to the branes, the radial direction r away from the branes, and some convenient set of T^{11} coordinates — e.g., the $\widehat{a}, \widehat{b}$ defined in the previous section.

It is natural then to solve (5.1) by separation of variables, expressing the dependence on the worldvolume directions in terms of plane waves $\exp(ik_\mu x^\mu)$, and expanding G and G_0 in terms of the scalar T^{11} harmonics as in (4.14),

$$G\left(r, \widehat{a}, \widehat{b}; r', \widehat{a}', \widehat{b}'\right) = \sum_{\mathbf{l},\mathbf{m}} Y^{\mathbf{lm}}_{(0)}\left(\widehat{a}, \widehat{b}\right) G^{(\mathbf{l})}(r, r') Y^{\mathbf{lm}}_{(0)}\left(\widehat{a}', \widehat{b}'\right). \tag{5.2}$$

Using this and the eigenvalue equation (4.15) in (5.1), the radial propagator for the $\mathbf{l} \equiv (l, j, s)$ dilaton partial wave on the three-brane background (2.2) can be seen to satisfy

$$\left(\frac{1}{r^5}\partial_r(r^5\partial_r) - \frac{H_0(\mathbf{l})}{r^2} + q^2 H(r)\right) G^{(\mathbf{l})}(k^\mu; r, r') = \frac{i\delta(r - r')}{r^5}, \tag{5.3}$$

where q is given by (4.4). The corresponding propagator on the Ricci-flat background (2.1), $G_0^{(\mathbf{l})}$, satisfies the same equation with the replacement $H(r) \to 1$.

As explained in [15] (see also [87, 88]), the homogeneous version of (5.1) can be related to Mathieu's equation. Following the procedure of [18], the propagator on the geometry side of the correspondence (with boundary conditions such that the associated flux moves *away* from the source) can then be worked out to be

$$G^{(\mathbf{l})}(k^\mu; r, r') = \frac{\pi}{4r^2 r'^2} H^{(1)}\left(\nu, \ln\left(\frac{r_>}{R}\right)\right)\left\{H^{(2)}\left(\nu, \ln\left(\frac{r_<}{R}\right)\right) + \frac{\chi - \frac{1}{\chi}}{\chi - \frac{1}{\eta^2\chi}} H^{(1)}\left(\nu, \ln\left(\frac{r_<}{R}\right)\right)\right\}, \tag{5.4}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and r'. We are adopting here the notation of [15, 18]: $H^{(1,2)}(\nu, z)$ are associated Mathieu functions of the third and fourth kind, respectively, ν (not to be confused with the constant υ introduced in (4.17)) is the 'Floquet exponent' defined in [15, 87], $\eta \equiv \exp(i\pi\nu)$, and $\chi \equiv \varphi(-\nu/2)/\varphi(\nu/2)$, with $\varphi(\pm\nu/2)$ two of the coefficients involved in the definition of Mathieu functions. For further characterization of ν, η and χ (which are all υ-dependent functions of qR), see (5.16) below, references [15, 87], and [18, appendix A].

Again following [18], the propagator in the worldvolume side of the correspondence can be expressed in terms of Hankel functions,

$$G_0^{(\mathbf{l})}(k^\mu; r, r') = \frac{\pi}{4r^2 r'^2} H_\upsilon^{(1)}(qr_>)\left\{H_\upsilon^{(2)}(qr_<) + H_\upsilon^{(1)}(qr_<)\right\}. \tag{5.5}$$

The linear combination inside the braces, $2J_\upsilon$, is singled out by the requirement that the propagator be regular at $r=0$. Our conclusion then is that the conifold propagators $G^{(\mathbf{l})}$ and $G_0^{(\mathbf{l})}$ have the same form as those obtained in [18] for the spherically-symmetric case, with the replacement $l+2 \to \upsilon$.

Now that we have determined the dilaton propagators $G^{(\mathbf{l})}$ and $G_0^{(\mathbf{l})}$, we can proceed to compute the two-point function of the gauge theory operators dual to the partial waves of the dilaton,

$$\Delta_2^{(\mathbf{l})}(q^2) \equiv \langle \mathcal{O}^{\mathbf{lm}}(k)\mathcal{O}^{\mathbf{lm}}(-k)\rangle\,. \tag{5.6}$$

Notice from the propagator equations (5.1) or (5.3) that we are working now with a *canonically* normalized dilaton. As in [18], this implies for the gauge theory operators an overall normalization $\mathcal{O}^{\mathbf{lm}}(x) \sim R^4\,\mathrm{Tr}[F^2\cdots]$ (see footnote 22).

The prescription of [18] to determine $\Delta_2^{(\mathbf{l})}$ is simply to equate the amplitudes obtained on both sides of the correspondence for dilaton propagation between two points far away from the branes. Following [18], and given the form of the coupling (4.18), for the regime under consideration this can be seen to amount to the statement that, in the limit $r, r' \to \infty$,

$$G^{(\mathbf{l})}(r,r') = G_0^{(\mathbf{l})}(r,r') + \left[G_0^{(\mathbf{l})}(r,r'')\left(\frac{R^2}{r''}\right)^{\upsilon-2}\right]_{r''=0}\Delta_2^{(\mathbf{l})}\left[\left(\frac{R^2}{r''}\right)^{\upsilon-2}G_0^{(\mathbf{l})}(r'',r')\right]_{r''=0}. \tag{5.7}$$

This in turn implies that

$$\Delta_2^{(\mathbf{l})} = \lim_{\substack{r,r'\to\infty \\ r''\to 0}} \frac{G^{(\mathbf{l})}(r,r') - G_0^{(\mathbf{l})}(r,r')}{G_0^{(\mathbf{l})}(r,r'')(R^2/r'')^{2\upsilon-4}G_0^{(\mathbf{l})}(r'',r')}\,. \tag{5.8}$$

As $r, r' \to \infty$, the Mathieu functions in (5.4) asymptote to Hankel functions, and the propagator on the three-brane can be shown to simplify to [18]

$$G^{(\mathbf{l})}(r,r') \to \frac{1}{2q}\left(\frac{1}{rr'}\right)^{5/2}\left\{e^{iq(r-r')} + \frac{\chi-\frac{1}{\chi}}{\chi-\frac{1}{\eta^2\chi}}e^{iq(r+r')+i2\theta}\right\}, \tag{5.9}$$

with $\theta = -\pi(2\nu+1)/4$. The Ricci-flat-space propagator similarly reduces to

$$G_0^{(\mathbf{l})}(r,r') \to \frac{1}{2q}\left(\frac{1}{rr'}\right)^{5/2}\left\{e^{iq(r-r')} + e^{iq(r+r')+i2\theta_0}\right\}, \tag{5.10}$$

with $\theta_0 = -\pi(2\upsilon+1)/4$.

For the denominator of (5.8), we also need the asymptotic form of $G_0^{(\mathbf{l})}$ as one of its arguments tends to infinity and the other to zero,

$$G_0^{(\mathbf{l})}(r,r'') \to \sqrt{\frac{\pi}{2q}}\frac{r^{-5/2}r''^{\upsilon-2}}{\Gamma(\upsilon+1)}\left(\frac{q}{2}\right)^{\upsilon}e^{i(qr+\theta_0)}\,. \tag{5.11}$$

Using (5.9)–(5.11) in (5.8), we see that all dependence on r, r', r'' cancels out, which is important in order for the limit (and consequently $\Delta_2^{(\mathbf{l})}$) to be well-defined. Our final result for the two-point function then follows as

$$\Delta_2^{(\mathbf{l})}(q^2) = -\frac{2^{2\upsilon}}{\pi}\left[\Gamma(\upsilon+1)\right]^2 R^{8-4\upsilon}q^{-2\upsilon}\left\{e^{i\pi\upsilon}\frac{\chi-\frac{1}{\chi}}{\eta\chi-\frac{1}{\eta\chi}} - 1\right\}. \tag{5.12}$$

An important check on the above result is to see whether it satisfies the optical theorem, i.e., whether it is correctly related to the absorption probability for the corresponding dilaton partial wave. As in [15, 18], the latter takes the form

$$P_{\rm abs} = 1 - \left| \frac{\chi - \frac{1}{\chi}}{\eta\chi - \frac{1}{\eta\chi}} \right|^2 . \tag{5.13}$$

Expanding the field operator for the $\mathbf{l}$-th partial wave according to

$$\phi^{(\mathbf{l})}(x;r) = \int_0^\infty dq\,\sqrt{q}\int_{-\infty}^{\infty}\frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k},q}} r^{-2}J_\upsilon(qr)\left[e^{ik\cdot x}a^{(\mathbf{l})}_{\vec{k},q} + e^{-ik\cdot x}a^{(\mathbf{l})\dagger}_{\vec{k},q}\right], \tag{5.14}$$

where $k^0 = \omega_{\vec{k},q} \equiv \sqrt{q^2+\vec{k}^2}$, using the form (4.18) for the dilaton-brane coupling, and following [18], one can derive the precise statement of the optical theorem on the worldvolume side of the correspondence,[23]

$$P_{\rm abs} = 2{\rm Re}\left(\frac{\pi R^{4\upsilon-8}q^{2\upsilon}\Delta_2^{(\mathbf{l})}}{2^{2\upsilon}[\Gamma(\upsilon+1)]^2}\right) - \left|\frac{\pi R^{4\upsilon-8}q^{2\upsilon}\Delta_2^{(\mathbf{l})}}{2^{2\upsilon}[\Gamma(\upsilon+1)]^2}\right|^2 . \tag{5.15}$$

And indeed, inserting our result (5.12) for $\Delta_2^{(\mathbf{l})}$ into (5.15), one can verify that the probability (5.13) is correctly reproduced.

The correlators (5.12) can be expanded in powers of $S \equiv (qR)^2$. To understand the analytic structure of this expansion, it is convenient to employ the results of [15, 87] to write

$$\nu(S) = \upsilon + \lambda(S^2)\,, \qquad \chi(S) = e^{-\nu(S)\ln S}\beta(S^2) \equiv e^{-\lambda(S)\ln S+\gamma(S)}\,, \tag{5.16}$$

where λ and β are (υ-dependent) functions of S^2 that are analytic in a neighborhood of $S=0$ (e^γ is then seen to be analytic only for integer υ).

Curiously, the case $\upsilon = 2$, which corresponds to the mode that is symmetric on T^{11} (i.e., $l=j=s=0$), is qualitatively different from all the rest. For this mode $\lambda \equiv i\mu$ and $\gamma \equiv i\alpha$ turn out to be purely imaginary (implying that $\eta \equiv \exp(i\pi\nu)$ is real and χ a pure phase) [15, 87]. As expected, in this case one finds the same result as for the spherically-symmetric mode in [18],

$$\Delta_2^{(\mathbf{0})}(S) \propto iS^{-2}\sinh(\pi\mu)\cot\left[-\mu\ln(-S)+\alpha\right] + S^{-2}\left(1-\cosh(\pi\mu)\right). \tag{5.17}$$

The analytic functions μ and α have leading behavior S^2 and S^0, respectively, so $\Delta_2^{(\mathbf{0})}$ has a sensible low-energy expansion [18],

$$\begin{aligned} i\Delta_2^{(\mathbf{0})}(S) \propto\; & S^2\ln(-S) + S^4\left[-\frac{1}{24}(\ln(-S))^2 + \frac{7}{72}\ln(-S)\right] + \\ & + S^6\left[\frac{17}{6912}(\ln(-S))^3 - \frac{161}{18432}(\ln(-S))^2 + \left(\frac{5\pi^2}{13824}+\frac{5561}{663552}\right)\ln(-S)\right] + \cdots . \end{aligned} \tag{5.18}$$

[23] Notice that equation (45) in [18] erroneously claimed that it was the imaginary part of Δ_2 that is relevant for the optical theorem. This error was however compensated by the fact that an i was incorrectly omitted from the right-hand side of equations (14) and (15) in that paper. As a result, the expressions for G, G_0 and Δ_2 given there were all missing a factor of i.

JHEP06(2003)049

Notice that the leading term in (5.18) is as dictated by conformal invariance for an operator of conformal dimension $\Delta = 4$. This confirms that, at low energies, our result correctly reduces to the expected two-point function in the Klebanov-Witten gauge theory. This is non-trivial because, as seen in (5.8), we have taken the limit $r, r' \to \infty$, i.e., we are probing the three-brane geometry from afar. The second term in (5.18) is the leading correction due to our departure from the conformal fixed point. As expected, $\Delta_2^{(\mathbf{0})}(S)$ has a branch cut for real positive S, which comes entirely from the first term in (5.17). As noted already in [18], the presence of the second term in (5.17) implies that for real negative S (i.e., away from the cut) the two-point function is *not* purely imaginary.[24] However, since the real terms are analytic, one is tempted to simply discard them.

For all $\upsilon > 2$, λ and γ turn out to be real (implying that η is a pure phase and χ is real) [15, 87]. Let us first consider the cases where $\upsilon \in \mathbb{Z}$. The results of [77, 78] reviewed at the end of the previous section imply that υ is an integer for *half* of the operators with protected dimensions. In more detail: for operators in short multiplets, we have $l = j = |s/2|$, and consequently $\upsilon = 3l + 2$; for operators in semi-long multiplets, $l = j - 1 = |s/2|$ (or $j = l - 1 = |s/2|$), and therefore $\upsilon = 3l + 4$ $(3j + 4)$. So in both cases $\upsilon \in \mathbb{Z}$ only if l and j are integer (rather than half-integer). Under these circumstances one finds that

$$\Delta_2^{(\mathbf{1})}(S) \propto iS^{-\upsilon} \sin(\pi\lambda) \coth\left[-\lambda \ln(-S) + \gamma\right] + S^{-\upsilon}\left(1 - \cos(\pi\lambda)\right) . \tag{5.19}$$

As in (5.17), we observe a clean separation between the imaginary and real parts. Notice, however, that the leading behavior of both terms is now of order $S^{2-\upsilon}$ (λ is still of order S^2, and γ is now of order $\ln(S)$), implying that $\Delta_2^{(\mathbf{1})}$ is *singular* in the IR. This important point was missed in the analysis of the spherically-symmetric case in [18], where υ becomes $l + 2$ and is therefore always integer. It is difficult to know what to make of this puzzling behavior. Taken at face value, the appearance of terms that are singular at low-energy is reminiscent of results obtained in the context of noncommutative theories, due to the phenomenon of UV/IR mixing [89, 90]. It would perhaps be worth exploring this possible connection (see [18] for a discussion of additional parallels with noncommutative theories).

In the general case $\upsilon \notin \mathbb{Z}$ (which covers the case of generic operators in long multiplets, as well as operators in protected multiplets with half-integer l and j), rewriting (5.12) in terms of the variables λ and γ defined in (5.16) one finds that $\Delta_2^{(\mathbf{1})}(S)$ becomes proportional to

$$S^{-\upsilon}\left\{e^{i\pi\upsilon}\frac{\cos(\pi\lambda)\sinh\left[-\lambda\ln(-S)+\gamma\right] - i\sin(\pi\lambda)\cosh\left[-\lambda\ln(-S)+\gamma\right]}{\cos(\pi\upsilon)\sinh\left[-\lambda\ln(-S)+\gamma\right] + i\sin(\pi\upsilon)\cosh\left[-\lambda\ln(-S)+\gamma\right]} - 1\right\} . \tag{5.20}$$

The leading low-energy behavior of this expression can again be shown to be singular, of order $S^{2-\upsilon}$, but the analytic structure is clearly much more intricate than in the $\upsilon \in \mathbb{Z}$ case.

[24] In [18] the two-point function was expected to be purely *real*, due to the missing factor of i mentioned in the previous footnote.

Notice in particular that there is now similar non-analytic behavior in both the imaginary *and* the real parts of $\Delta_2^{(1)}$. By applying the recipe of [18] in the conifold setting, we have thus learned that in general it would be a mistake to simply discard the unwanted real terms in $\Delta_2^{(1)}$, as one was tempted to do in the spherically-symmetric situation. In any event, as was emphasized in [18], it is the entire expression (5.12) that is needed in order to satisfy the optical theorem (5.15).

The expression (5.12) (or (5.20)) can be expanded in powers of S, to find a series of the general form

$$\Delta_2^{(1)}(S) = -\frac{2^{2v}}{\pi}\left[\Gamma(v+1)\right]^2 R^{8-2v} \sum_{n,m=0}^{\infty}\sum_{k=0}^{m} C_{nmk}(v) S^{(2n-1)v+2m} \left(\log(-S)\right)^k + \text{analytic}\,. \tag{5.21}$$

As we have just noted, the coefficients C_{nmk} in this expansion are in general *complex* (see, e.g., the explicit expressions in (5.23) below). The reason for this can be understood by looking back at (5.7). If we had worked in momentum space for all ten of the spacetime directions, then the propagators $G^{(1)}$ and $G_0^{(1)}$ would have been as usual purely imaginary, and so (5.7) would imply that $\Delta_2^{(1)}$ is purely imaginary as well. We are working in momentum space for the directions parallel to the branes, but in position space for the transverse directions, and as a result, the propagators (5.4) and (5.5) are manifestly complex. The definition (5.7) then implies the same for $\Delta_2^{(1)}$. Notice, however, that our two-point function does meet the physical requirement of unitarity, as expressed in the optical theorem (5.15). One should also bear in mind that the appearance of complex coefficients in correlation functions is not unique to the approach of [18]: it is also seen, e.g., in the two-point functions derived in [20, 21] (a fact that was not noticed in those works). For additional discussion of this issue, see [18].

Notice that (5.21) reads

$$i\Delta_2^{(1)}(S) \propto \text{IR-singular terms} + C_{100}S^v + C_{111}S^{v+2}\ln(-S) + \text{subleading terms}\,. \tag{5.22}$$

The term of order S^v is of the form expected by conformal invariance for the two-point function of an operator of conformal dimension $\Delta = 2+v$ in the Klebanov-Witten gauge theory, but now due to the IR singularities this is *not* the dominant term in $\Delta_2^{(1)}$ at low energies. The S^{v+2} term shown above (as well as a similar term without the logarithm) has the form required for the leading correction due to our departure from the conformal fixed point. For the spherically-symmetric case, it was shown in [24] that the analogous term could be reproduced via an AdS/CFT three-point function computation. It would be very interesting to check if the same can be done here, using the standard GKPW recipe [10, 11] in the $\mathrm{AdS}_5\times T^{11}$ setting to work out the three-point function $\langle\mathcal{O}^{\mathbf{lm}}\mathcal{O}^{\mathbf{lm}}\mathcal{O}_8\rangle$, obtained by inserting in $\Delta_2^{(1)}$ the dimension-eight operator (3.5) coming from the deformed worldvolume lagrangian (3.4). Evidently more work is needed also to clarify the meaning of the IR singularities observed in $\Delta_2^{(1)}$, and in particular whether they are somehow an artifact of the recipe of [18].

With these future applications in mind, we record here the first few of the coefficients in the expansion (5.21), worked out for the case $v \notin \mathbb{Z}$ using `Mathematica`:

$$
\begin{aligned}
C_{000} &= 0 \\
C_{010} &= -\frac{i\pi}{4v - 4v^3} \\
C_{011} &= 0 \\
C_{020} &= -\frac{\pi^2}{2(4v - 4v^3)^2} + \frac{i\pi(8 - 35v^2 + 15v^4)}{64v^3(-4 + v^2)(-1 + v^2)^3} \\
C_{100} &= -\frac{2^{1-4v}\pi^2}{\Gamma(v)^2\Gamma(1+v)^2} + \frac{i4^{-2v}\pi^2 \csc^2(\pi v)\sin(2\pi v)}{\Gamma(v)^2\Gamma(1+v)^2} \\
C_{110} &= \frac{2^{-1-6v}\pi^3 \csc(\pi v)[4^v \cos(\pi v)\Gamma(v)\Gamma(1+v) + 8v(-1+v^2)k_2]}{v(-1+v^2)\Gamma(v)^3\Gamma(1+v)^3} - \\
&\quad - \frac{i2^{-1-6v}\pi^3 \cot(\pi v)\csc(\pi v)[4^v \cos(\pi v)\Gamma(v)\Gamma(1+v) + 8v(-1+v^2)k_2]}{v(-1+v^2)\Gamma(v)^3\Gamma(1+v)^3} \\
k_2 &\equiv \frac{4^v \cos(\pi v)\Gamma(v)\Gamma(1+v)}{4v - 4v^3} + \frac{1}{\pi v^2(-1+v^2)^2} \times \\
&\quad \times \Bigg(4^{-1+v}\Gamma(v)\Gamma(1+v)\sin(\pi v)\Bigg[1 + (v - v^2)[v(-1 + \ln 4) + \ln 4] - \\
&\qquad\qquad - 2v(-1+v^2)\frac{\Gamma'(v)}{\Gamma(v)}\Bigg]\Bigg) \\
C_{111} &= -\frac{4^{1-2v}\pi^2}{(4v - 4v^3)\Gamma(v)^2\Gamma(1+v)^2} + \frac{i2^{1-4v}\pi^2\csc^2(\pi v)\sin(2\pi v)}{(4v - 4v^3)\Gamma(v)^2\Gamma(1+v)^2} \\
C_{122} &= -\frac{4^{1-2v}\pi^2 \csc(\pi v)(-i\cos(\pi v) + \sin(\pi v))}{(4v - 4v^3)^2\Gamma(v)^2\Gamma(1+v)^2}\,.
\end{aligned}
\tag{5.23}
$$

6. Conclusions

Building upon previous work [6], [13]–[16] and [18], in this paper we have studied the higher-energy precursor of the AdS/CFT correspondence, concentrating for the most part on the $\mathcal{N} = 1$ duality that equates the two alternative descriptions of a stack of D3-branes living on the conifold [31]. In section 2.1 we have discussed some of the issues that arise in trying to formulate a precise duality statement at the level of string theory (prior to taking any limits). Our main point has been that the difficulties one encounters are not limitations of principle, but of practice — not unlike what one runs into in the various instances of S-duality in string theory, or, to some extent, in the AdS/CFT case itself.

In section 2.2 we have reviewed the various ways in which the above 'worldvolume/geometry' string duality can be cut down to more accessible sizes. These include in particular the Maldacena limit $l_s \to 0$ with R/l_s fixed [9], and the Klebanov limit $l_s \to 0$ with R fixed [6]. An important point is that, from the outset, the simplification process necessarily involves the introduction of a cutoff $\Lambda < 1/l_s$ on both sides of the correspondence. Of course, in the Maldacena case, *after* flowing down to the conformal fixed point one can choose to remove the cutoff; but in the general case the presence of the cutoff is crucial in order to make sense of the duality, because the field theories one equates are ineluctably non-renormalizable.

Along the way, we have seen in equation (2.5) that the well-known UV-IR connection ultimately arises from the redshift factor of the three-brane metric (2.2). It would be interesting to try to relate this discussion to the curious form of the UV-IR relation deduced for the full three-brane background (in the flat-space, as opposed to conifold, setting) in [18]. It was shown there that the profile of a one-point function in the presence of a source localized in the bulk at first decreases in size as the source moves to larger values of r, but then starts to grow again when the source has moved past $r = R$.[25] In terms of the discussion of section 2.2 it is in a sense natural for the asymptotically (Ricci-)flat region $r \to \infty$ (where the massless supergravity modes are always present) to correspond to low energies, but it would be worth trying to further elucidate the full picture.

As explained in section 2.2, as long as $g_s N \gg 1$ there is a wide intermediate-energy region $0 < \Lambda R < (g_s N)^{1/4}$ where one stays away from the AdS/CFT endpoint $\Lambda R = 0$ (which implies in particular that the branes remain coupled to the bulk), and yet one is still able to neglect — modulo redshift effects — the massive string modes (which means that the duality can be formulated in essentially field-theoretic terms). From section 3 on we chose to restrict our analysis to the Klebanov limit [6], $l_s \to 0$ with R fixed (implying $g_s N \to \infty$), which places us at a particularly tractable corner of the full parameter space available in this intermediate-energy region.

In this particular limit, one confronts a duality that equates free supergravity on the conifold-three-brane background (2.2) with the intermediate-energy effective action on the D3-brane worldvolume, *coupled to* free supergravity in the ten-dimensional Ricci-flat geometry (2.1) [6, 14, 13, 18]. In section 3, by adapting the symmetry arguments of [15, 16] to our setting, and employing the results of [77, 78] for the $\mathrm{AdS}_5 \times T^{11}$ spectrum, we have been able to conjecture the specific form (3.4) for the required worldvolume action.

In section 4 we have reviewed the way in which the various dilaton partial waves $\phi^{(l,j,s)}$ couple at linear order to the worldvolume action, as well as the form of the gauge theory operators $\mathcal{O}^{l,j,s}$ dual to them [77, 78]. We have then proceeded in section 5 to work out the two-point functions for these operators, using the recipe of [18, 22]. Just like in the case of the asymptotically flat three-brane background [15], the dilaton equation of motion on the conifold-three-brane background can be related to Mathieu's equation, which allows an analytic determination of the corresponding propagator, and consequently, of the gauge-theory two-point function. Our final result is given in (5.12).

Following [18], we have shown that the two-point functions we obtain are correctly related by the optical theorem, equation (5.15), to the absorption probability for the corresponding dilaton partial wave. For the mode that is symmetric on T^{11}, a further check on our result is that, as seen in (5.18), at low energies our two-point function reduces to the one expected in the context of the Klebanov-Witten gauge/gravity duality. For higher partial waves, it can be seen in (5.22) that one also finds a term with the required Klebanov-Witten behavior, but the leading low-energy behavior is in fact *singular*. This

[25] The same type of behavior was obtained in the context of the supergravity dual of NCYM in [91], using a recipe for correlation functions [22] that is identical to that of [18].

applies both to the conifold and the flat-space cases (an important point that was missed in [18]). More work is clearly needed to determine whether this behavior is real or spurious, and in the former case, to identify its physical origin.

Equations (5.18) and (5.22) show also the leading correction to the two-point functions due to our departure from the conformal fixed point. As mentioned in section 5, it would be very interesting to try to test the mutual consistency of our two main results, (3.4) and (5.12), by showing that this leading correction matches the pure $\mathrm{AdS}_5 \times T^{11}$ three-point function $\langle \mathcal{O}^{l,j,s}\mathcal{O}^{l,j,s}\mathcal{O}_8 \rangle$, with $\mathcal{O}_8$ the dimension-eight operator (3.5) coming from the deformed worldvolume lagrangian (3.4). For the spherically-symmetric case, the agreement between the two analogous quantities was demonstrated in [24].

As in the asymptotically flat case, we have seen here, in the conifold setting, that the recipe of [18, 22] yields correlation functions that are automatically finite, despite the fact that in (5.8) one takes r and r' (which according to standard AdS/CFT intuition would be associated with a UV cutoff) to infinity. This is unlike the situation encountered in applications of the standard GKPW recipe [10, 11] (which in some cases can even require momentum-dependent 'renormalization', as in [20, 21]). An important pending task is to try to extract from these results information about the precise way in which the cutoff Λ is implemented on the 'worldvolume' side of the duality.

Obvious extensions of our work would be to determine the two-point functions of operators dual to other supergravity fields, or to work out higher n-point functions. One could also apply the same methods to explore analogous intermediate-energy dualities in other interesting backgrounds, like the warped deformed conifold of [92].[26]

Clearly much work remains to be done to understand the precise nature of this 'worldvolume/geometry' duality at the string theory level. We would like to return in particular to the subject of the relation between the duality and the Fischler-Susskind mechanism, which we brought up in section 2.1. Further progress could perhaps also be made by looking for settings where the duality can be studied in simplified but still string-theoretic terms. We believe that a particularly promising example of this is the Nonrelativistic Wound string theory setting [93, 94, 95], where the 'worldvolume' description of a stack of (longitudinal) Dp-branes is provided by NCOS theory [96, 97], and, as explained in [98] (see also [99]), the 'geometry' description of the stack is afforded by the dual supergravity backgrounds obtained in [97, 100].

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[26] Unfortunately, in this new setting the dilaton equation of motion is no longer related to Mathieu's equation, and so one can no longer obtain the complete solution in analytic form.

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